Reprinted from JOURNAL OF PHYSICAL OCEANOGRAPHY, Vol. 16, No. 12, December 1986
American Meteorological Society

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(Manuscript received 15 July 1985, in final form 21 May 1986)

ABSTRACT

A general maximum entropy estimate for the directional distribution of the directional wave spectrum is introduced. The estimate appears to be superior to conventional methods, including maximum likelihood methods, when applied to data from a heave/pitch/roll buoy. The simplicity of the estimate makes it easy to apply when an expression for the directional distribution rather than its moments is sought. The estimate is illustrated with wave data from the Norwegian Sea.

1. Introduction

There are a variety of systems for wave directional measurements which produce estimates of the Fourier coefficients of the directional distribution. From point measurements, such as heave/pitch/roll buoys and 3-axial current meters, the first four coefficients may be obtained, whereas arrays and curvature measuring devices may produce estimates of additional coefficients. Even if the lowest-order Fourier coefficients yield interesting information such as the mean direction and the spreading of the directional distribution, there still remains the question of obtaining a reasonable estimate for the distribution itself. For arrays, the maximum likelihood method (MLM) has received some attention (Baggeroer, 1979; Borgman, 1982; Isobe et al., 1984), whereas for buoys, the most common approach has been to fit some simple analytical distribution to the mean direction and spreading. The possibility of using maximum entropy estimation techniques has previously been suggested by Borgman (1982). Below we consider a maximum entropy method (MEM) for obtaining a directional distribution from estimates of the Fourier coefficients by utilizing the analogy with spectral estimation for complex Gaussian, stationary processes. The latter problem has received great attention for real processes, and the generalization to complex processes is straightforward. Unlike other directional estimates, the MEM estimate reproduces exactly all Fourier coefficients that are input to the estimate. Finally, we show examples of MEM estimates of the directional distribution based on wave data from a heave/pitch/roll buoy operated offshore of Norway. The MEM estimate is seen to be capable of resolving two wave fields coexisting in the same frequency band, and the results are given physical confirmation from surface weather charts.

2. Theory

Let $f(\theta)$ be a positive function on $(-\pi, \pi)$ with integral equal to 1 and Fourier series

$$f(\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n{}^{in\theta}, \quad c_0 = 1, \quad c_{-n} = c_n^* \tag{1}$$

where the asterisk denotes complex conjugate.

The entropy of f is the functional defined by

$$H(f) = -\int_{-\pi}^{\pi} \log(f(\theta))d\theta. \tag{2}$$

It may be shown (Burg, 1975; Ulrych and Bishop, 1975) that the positive function maximizing $H(f)$ subject to the constraints

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta = c_k, \quad k \leqslant N \tag{3}$$

is

$$f(\theta) = \frac{1}{2\pi}\frac{\sigma_e{}^2}{|1 - \phi_1 e^{-i\theta} - \cdots - \phi_N e^{-iN\theta}|^2} \tag{4}$$

where the parameters $\phi_1 \cdots \phi_N$ and $\sigma_e{}^2$ are obtained from the Yule–Walker (Y–W) equations:

$$\begin{bmatrix} 1 & c_1^* & \cdots & c_{N-1}^* \\ c_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1^* \\ c_{N-1} & \cdots & c_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \phi_N \end{bmatrix} = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_N \end{bmatrix} \tag{5}$$

$$\sigma_e{}^2 = 1 - \phi_1 c_1^* - \cdots - \phi_N c_N^*. \tag{6}$$

The function $f(\theta)$ defined in (4) is the MEM estimate based on $c_1, \ldots, c_N$. This function is also the spectral density of a complex autoregressive process $\{Z_k\}$ of the form $Z_k - \phi_1 Z_{k-1} - \cdots - \phi_N Z_{k-N} = e_n$, where $\{e_n\}$ is white noise (Burg, 1975). Denote the coefficient matrix in (5) by R_{N-1}. The sequence, $1, c_1, c_2, \ldots$, $(c_{-j} = c_j^*)$, is the Fourier coefficients of a positive function if and only if it is positive semidefinite (Hannan, 1970). This is characterized by $\det(R_N) \geqslant 0$, $N = 0, 1, 2, \cdots$, or, equivalently, R_N being positive semidefinite for $N \geqslant 0$. It is always possible to extend $\{c_N^*, \ldots, c_1^*, 1, c_1, \ldots, c_N\}$ to a doubly infinite positive-definite sequence provided $\det(R_k) \geqslant 0$ for $k \leqslant N$. In fact, if $\det(R_i) > 0$ for $i = 0, \cdots, N-1$, and $\det(R_N) \geqslant 0$, then $\det(R_{N+1}) \geqslant 0$ if and only if

$$|c_{N+1} - \gamma_c \phi|^2 \leqslant (1 - \gamma^H \phi)\det(R_N)/\det(R_{N-1}) \quad (7)$$

where $\gamma = (c_1, \cdots, c_N)^t$, $\gamma_c = (c_N, \cdots, c_1)$, $\phi = (\phi_1, \cdots, \phi_N)^t$, $R_{N-1}\phi = \gamma$ and $\gamma^H = (\gamma^t)^*$. Thus, on step "$N+1$", c_{N+1} has to be chosen from a disc in the complex plane defined by (7). It may be proven that the MEM estimate corresponds to picking the center of the disc. This, incidentally, also maximizes $\det(R_{N+1})$. The MEM estimate is therefore a most natural choice, and we refer to Burg (1975) for a more elaborate treatment of these properties.

The maximum likelihood (MLM) estimate of $f(\theta)$ based on $1, c_1, \ldots, c_N$ is defined

$$f_{\mathrm{MLM}}(\theta) = \kappa / h(\theta)^H (R_N)^{-1} h(\theta) \quad (8)$$

where $h = (1, e^{i\theta}, \cdots, e^{iN\theta})^t$ and κ is a normalizing constant (Isobe et al., 1984). As long as the estimate, $\hat{R}_N$, of R_N is Hermitian and positive definite, $h^H \hat{R}_N^{-1} h$ will be a positive trigonometric polynomial of order N, and as such it has a representation of the form

$$|\phi_0 - \phi_1 e^{-i\theta} - \cdots - \phi_N e^{-iN\theta}|^2$$

(Grenander and Szegö, 1958). Thus, both the MLM and the MEM estimates have the form displayed in (4). However, the Fourier coefficients of f_{MLM} are, in general, different from $\{\ldots, 1, c_1, \ldots, c_N, \ldots\}$.

3. The MEM estimate for heave/pitch/roll data

Let the ideal output from a heave/pitch/roll buoy be $X(t) = \{X_1(t), X_2(t), X_3(t)\}$, where subscript "1" denotes heave, "2" slope in the east direction, and "3" slope in the north direction. The following identities between the cross-spectral matrix of X, $\{C_{ij}(\omega) - iQ_{ij}(\omega)\}$, and the directional wave spectrum $E(\omega, \theta) = S(\omega)D(\theta, \omega)$ were obtained by Longuet-Higgins (1963):

$$\left.\begin{aligned}
C_{12}(\omega) &= C_{13}(\omega) = Q_{23}(\omega) = 0 \\
C_{11}(\omega) &= S(\omega)\int_{-\pi}^{\pi} D(\theta, \omega)d\theta \\
C_{22}(\omega) &= k^2 S(\omega)\int_{-\pi}^{\pi} \cos^2(\theta)D(\theta, \omega)d\theta \\
C_{33}(\omega) &= k^2 S(\omega)\int_{-\pi}^{\pi} \sin^2(\theta)D(\theta, \omega)d\theta \\
Q_{12}(\omega) &= kS(\omega)\int_{-\pi}^{\pi} \cos(\theta)D(\theta, \omega)d\theta \\
Q_{13}(\omega) &= kS(\omega)\int_{-\pi}^{\pi} \sin(\theta)D(\theta, \omega)d\omega \\
Q_{23}(\omega) &= k^2 S(\omega)\int_{-\pi}^{\pi} \cos(\theta)\sin(\theta)D(\theta, \omega)d\omega \\
k &= k(\omega), \quad \text{the dispersion relation}
\end{aligned}\right\}. \quad (9)$$

In practice, estimates of C and Q are formed by

$$C(\omega) - iQ(\omega) = \Sigma_{\omega'} X^*(\omega')X(\omega')w(\omega - \omega')$$

where X^* is the discrete Fourier transform of X and w is a positive weight function centered around 0. Following Long (1980), we base the estimates of the Fourier coefficients of $D(\theta)$ on the relations

$$\left.\begin{aligned}
\hat{d}_1 &= Q_{12}/[C_{11}(C_{22} + C_{23})]^{1/2} \\
\hat{d}_2 &= Q_{13}/[C_{11}(C_{22} + C_{33})]^{1/2} \\
\hat{d}_3 &= (C_{22} - C_{33})/(C_{22} + C_{33}) \\
\hat{d}_4 &= 2C_{23}/(C_{22} + C_{33}), \\
\hat{c}_1 &= \hat{d}_1 + i\hat{d}_2 \\
\hat{c}_2 &= \hat{d}_3 + i\hat{d}_4
\end{aligned}\right\}. \quad (10)$$

(We have suppressed the dependence on ω).

This definition has the advantage of being rather insensitive to errors in the buoy calibration, and further calculations reveal that the matrix

$$\begin{bmatrix} 1 & \hat{c}_1^* & \hat{c}_2^* \\ \hat{c}_1 & 1 & \hat{c}_1^* \\ \hat{c}_2 & \hat{c}_1 & 1 \end{bmatrix} \quad (11)$$

is *necessarily positive definite.* This guarantees, by the general extension procedure discussed in section 2, the existence of a positive function $\hat{D}(\theta)$ such that

$$2\pi\hat{D}(\theta) = \cdots + \hat{c}_2^* e^{-2i\theta} + \hat{c}_1^* e^{-i\theta} + 1 + \hat{c}_1 e^{i\theta} + \hat{c}_2 e^{2i\theta} + \cdots. \quad (12)$$

Other expressions for the Fourier coefficients, e.g., using the theoretical rather than the measured dispersion relation, do not have this property (Barstow and Krogstad 1984).

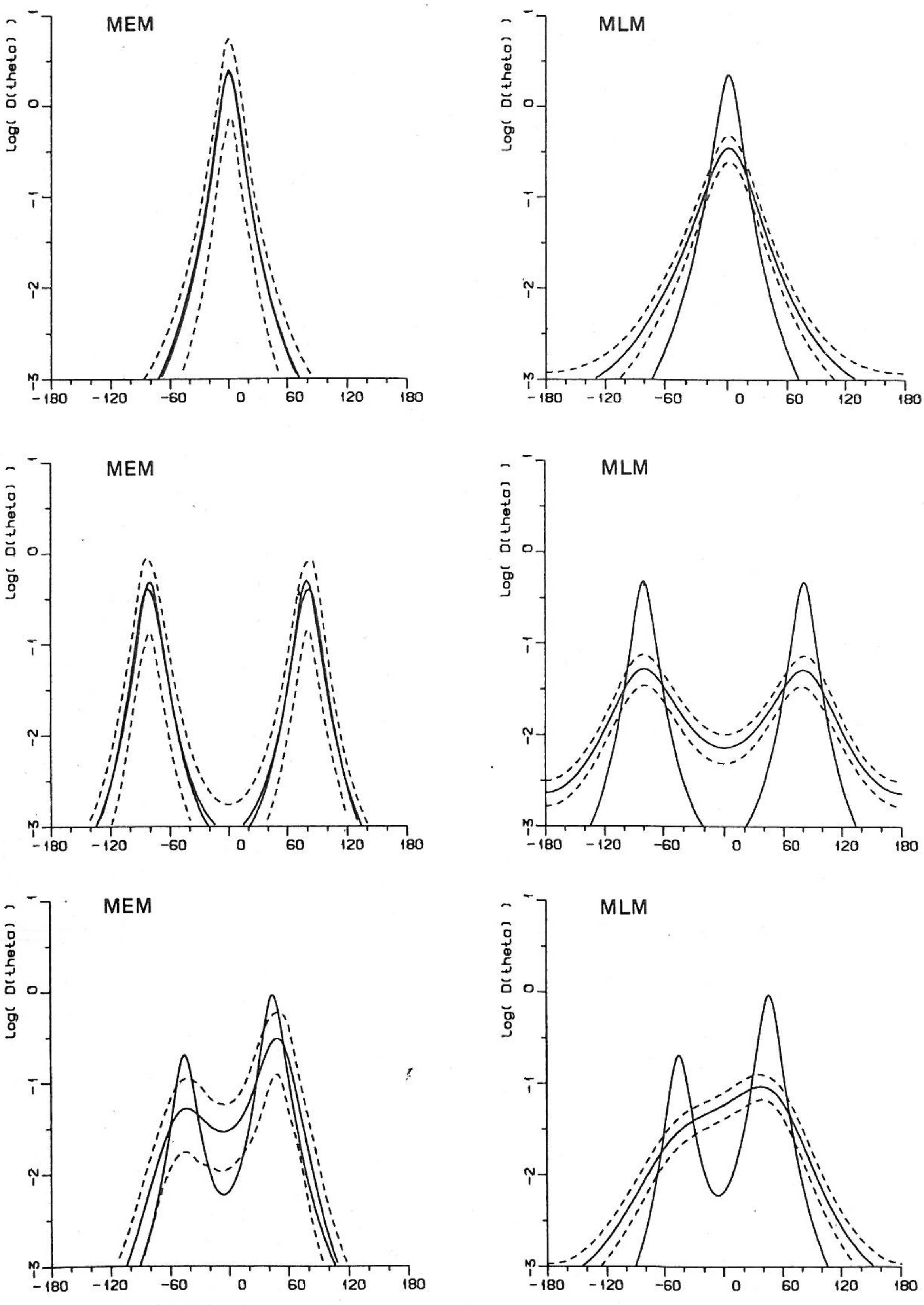


FIG. 1. MEM and MLM performance using simulated data. Mean value and ± one standard deviation of 100 outcomes are shown with 64 degrees of freedom in the cross-spectral estimates.

The MEM estimate of D now simply consists of inserting $\hat{c}_1$ and $\hat{c}_2$ into (5) and solving for ϕ_1 and ϕ_2. The estimate is then found from (4):

$$\left.\begin{aligned}\phi_1 &= (\hat{c}_1 - \hat{c}_2\hat{c}_1^*)/(1-|\hat{c}_1|^2)\\ \phi_2 &= \hat{c}_2 - \hat{c}_1\phi_1\\ 2\pi D(\theta) &= (1-\phi_1\hat{c}_1^* - \phi_2\hat{c}_2^*)/|1-\phi_1 e^{-i\theta} - \phi_2 e^{2i\theta}|^2\end{aligned}\right\}. \quad (13)$$

Any other configuration of wave sensors producing estimates of $\hat{d}_1, \ldots, \hat{d}_4$ may use the same expression for $\hat{D}$. The approximate statistics of D might be obtained from the statistics of the d-parameters by the Taylor expansion procedure used by Long (1980). An alternate approach is discussed in Baggeroer (1976). As the expressions get algebraically complicated, we shall only show the results from a series of simulations of $\hat{d}_1, \ldots, \hat{d}_4$ based on known directional distributions. The parameters have been obtained from the simulated cross spectra, i.e., multivariate, complex Gaussian random numbers with a known covariance matrix. The MLM estimate is shown for comparison. In Fig. 1, the mean

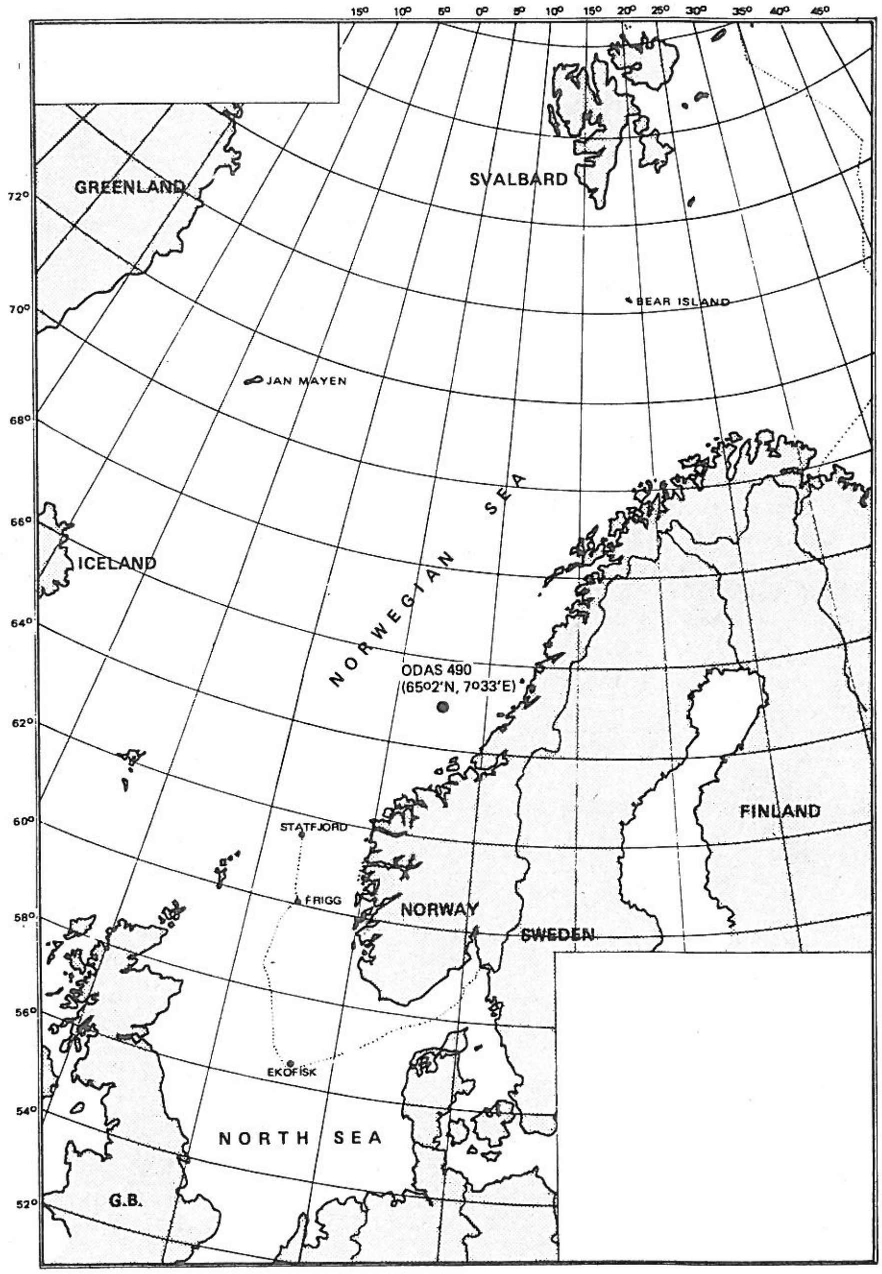


FIG. 2. Geographic position for the directional wave measurements at Haltenbanken.

± one standard deviation is displayed along with the following input theoretical distributions:

(a) $p(\theta, 0.8)$

(b) $0.5p(\theta-80°, 0.8)+0.5p(\theta+80°, 0.8)$

(c) $0.33p(\theta+45°, 0.8)+0.67p(\theta-45°, 0.8)$

where $2\pi p(\theta, x) = (1 - x^2)/(1 - 2x \cos\theta + x^2)$.

The choice favors the MEM estimate which has virtually no bias in cases (a) and (b). None of the methods are able to reproduce (c), but the MEM estimate has less bias. The MEM estimate thus seems to have less bias, but at the expense of a somewhat larger variance than the MLM estimate. This agrees well with what is known from MEM and MLM spectral estimation. The MEM estimate based on c_1 and c_2 from a boxcar directional distribution will show two peaks. This tendency to split peaks is often noted in MEM spectral estimation. Double peaks have been observed in a few cases for real data, but the lack of better measurements of the directional distribution prevents us from deciding whether the two peaks are real or simply an artifact of the estimate.

4. Directional wave measurements from the Norwegian sea

Routine wave directional measurements using NORWAVE heave/pitch/roll buoys have taken place on Haltenbanken on the Norwegian Continental Shelf since 1980 (Fig. 2). These measurements are carried out as a part of the Oceanographic Data Acquisition Project (ODAP), funded by the oil industry and Norwegian governmental authorities.

The NORWAVE buoy records time series of heave, pitch and roll as well as compass heading, mean wind speed and direction 4.2 m above the instantaneous water level, air and sea temperature and air pressure at a recording interval of 3 hours. An ARGOS satellite transmitter is used for buoy surveillance. The buoy design, reflecting a compromise between stability and wave-following capabilities, is not optimal for directional wave measurements, but the results so far have been promising. The buoy has a pitch/roll resonance with a period of around 2.8 sec, and the transfer functions for pitch and roll are found from the phase behavior of the cross spectra between heave and the slopes, assuming that the buoy behaves like a forced linear oscillator. NORWAVE data buoys are now in their fifth year of routine operations. During this period the buoys have survived under severe conditions with mean wind speeds over 25 m s^{-1} and significant wave heights up to 14 m with extreme waves over 20 m. A more detailed description of the buoy, its calibration and sensor intercalibrations are reported in Barstow et al. (1983), Audunson et al. (1982) and Barstow and Krogstad (1984). All data from the buoys are routinely processed and reported. The directional analysis is a conventional cross-spectral analysis producing estimates of $d_1, \ldots, d_4$ as discussed in section 3.

5. MEM directional distributions

Some results using data from the ODAP measurements are shown below. The MEM estimate has been obtained from smoothed cross spectra with 64 degrees of freedom. The time series length is 1024 s with a sampling frequency equal to 1 Hz.

Figure 3 shows the time series of wind speed and wind direction as recorded by the NORWAVE buoy on 9–11 September 1980 at Haltenbanken. A steplike change in wind direction occurs on the 10th, and it is of interest to see whether the MEM estimate can resolve the expected double-peaked nature of the directional distribution during this event. The directional distribution at 0.35 Hz in the wind-wave range of the spectrum is shown for 3 hour intervals in Fig. 4. The peak in the directional distribution close to the old wind direction steadily decays and disappears completely 9 hours after the initial change in the wind direction. Meanwhile, a new peak rises associated with the new wind direction. It is clear that the MEM-fit reproduces these features better than does the MLM-fit in that the latter tends to smooth the expected two-peaked nature of the directional distribution.

Figure 5 shows surface weather charts at 0600 GMT 9 and 10 September. The pressure field is more or less the same both days with a depression to the north of Norway moving slowly northeastwards. Northwesterly winds dominate in the Norwegian Sea northwest of the buoy position, while locally strong curvature in the isobars along the line of the trough explains the rapid change in wind direction from southwest to northwest on the 10th. The contour plots of the directional spectrum at 3-hour intervals late on the 10th show a distinct

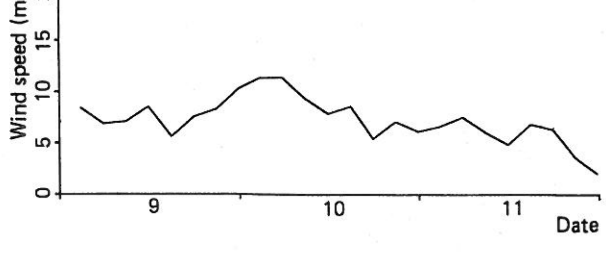


FIG. 3. Wind speed and direction recorded by ODAS 490 on 9–11 September 1980.

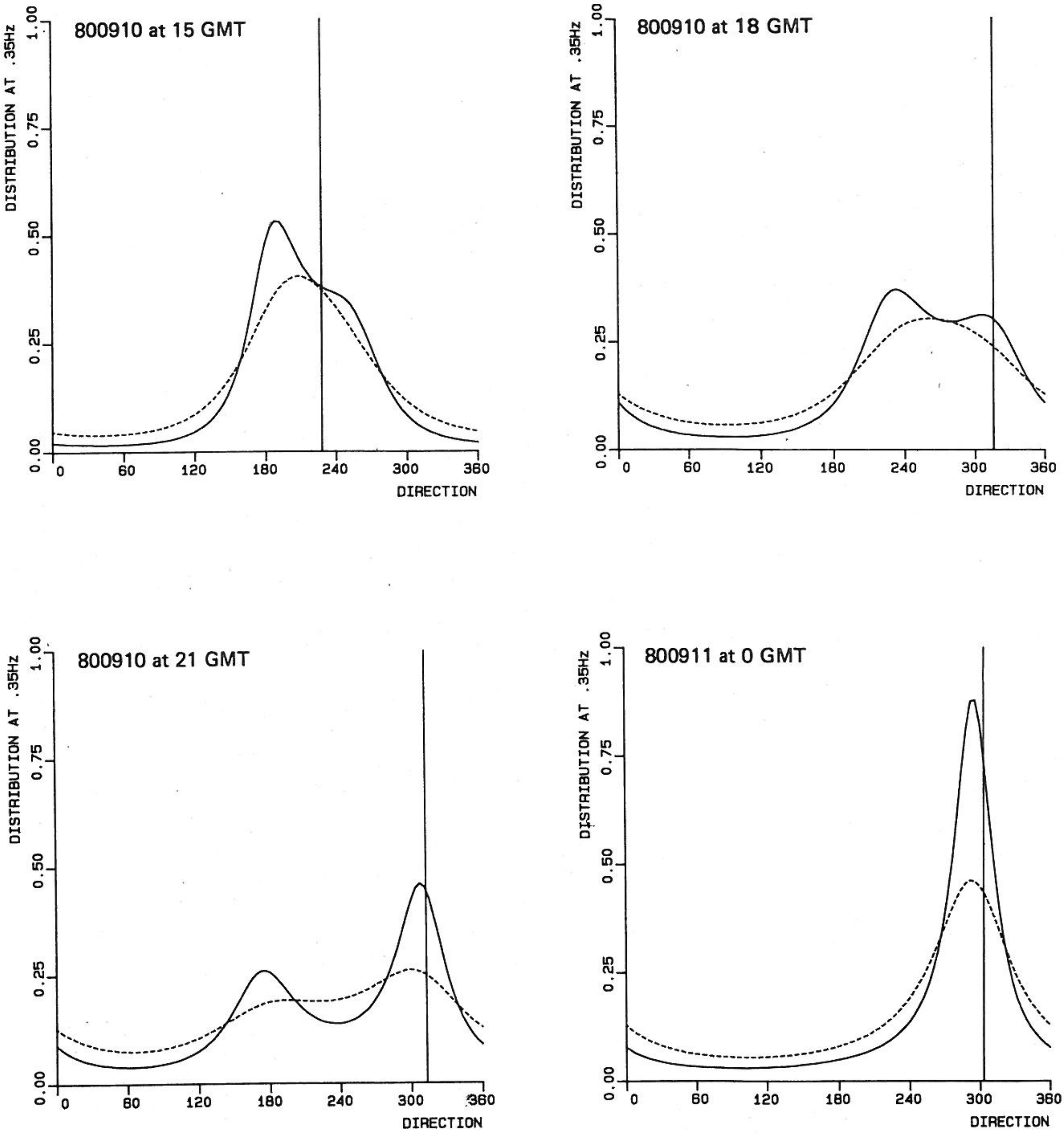


FIG. 4. MEM estimate for the directional distribution at 0.35 Hz during the veering wind. The wind direction is shown by a vertical line. Solid line: MEM; broken line: MLM.

two-peaked directional distribution around the spectral peak (Fig. 6). The bimodality of the wave field persists for at least 6 hours. The waves coming from northwest are well aligned with the wind direction and apparently generated by the northwesterly winds in the Norwegian Sea. The sudden appearance and gradual decrease in energy for the wave field coming from south-southwest (around 200°) are typical for swell waves propagating into the area in question. The surface weather charts displayed in Fig. 5 do not provide details of the wind field sufficient to explain the latter wave field. However, we know from the wind direction observations that the wind field was relatively nonhomogeneous (cf. Fig. 3), and the synoptic situation also gives some evidence for southwesterly winds in the area to the south of the buoy early on the 10th. Waves coming from south-

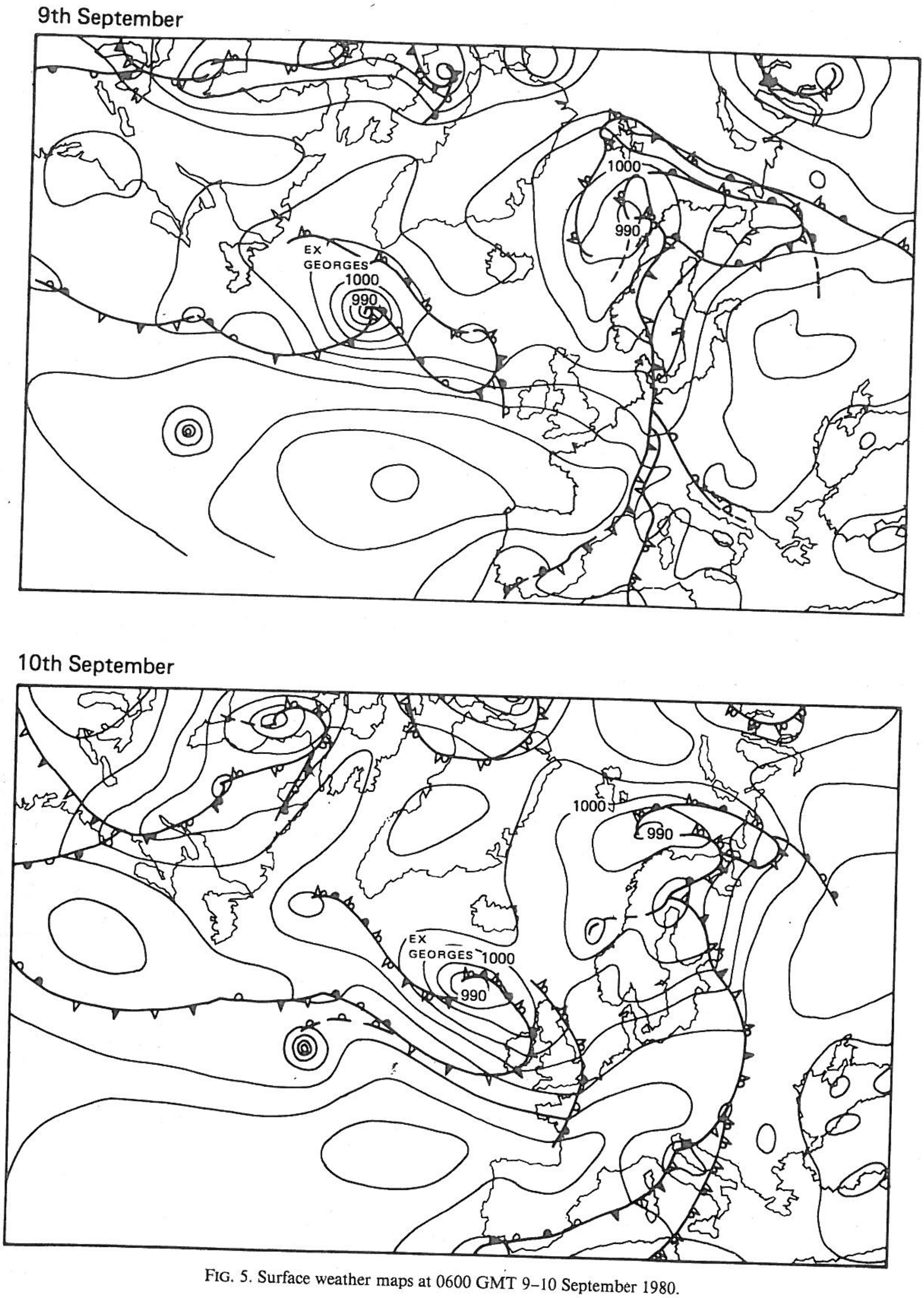


FIG. 5. Surface weather maps at 0600 GMT 9–10 September 1980.

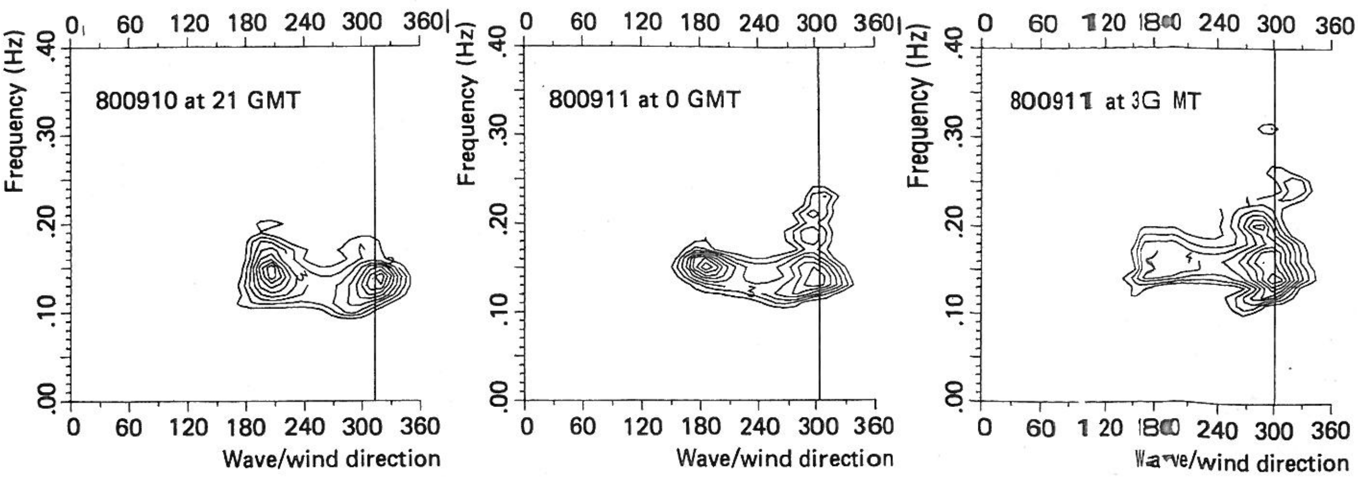
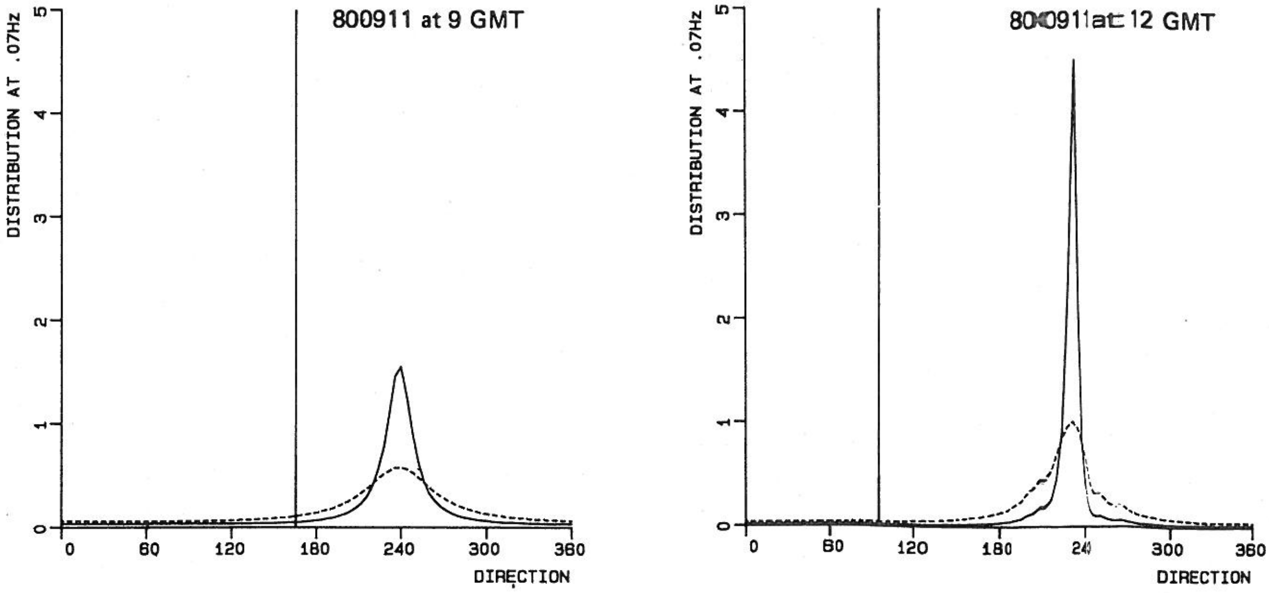


FIG. 6. Contour plot of the directional wave spectrum, 10–11 September 1980. Contour interval: 1.4 dB.



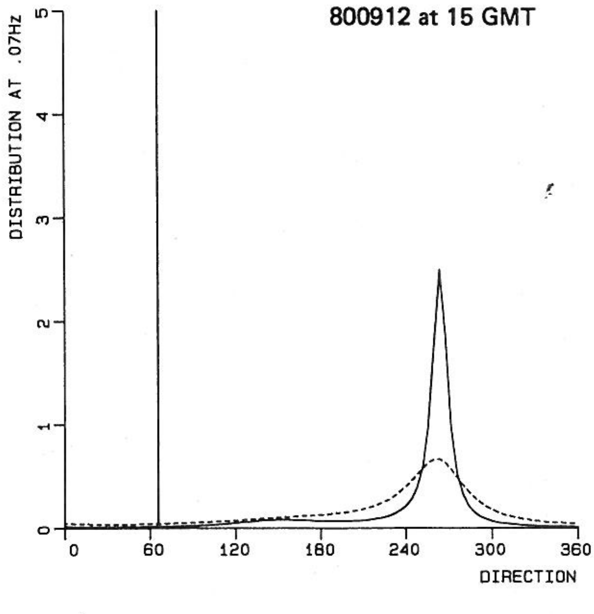


FIG. 7. Recorded swell at 0.07 Hz, 11 September 1980. Local wind direction also shown.

southwest are fetch limited, and the JONSWAP relations stated in Hasselmann et al. (1973) support generation of waves at the observed peak frequency for fetches of order 200 km and wind speeds around 10 m s^{-1}. This agrees well with the actual fetches and wind speed observations made along the coast.

Swell from the Ex-Georges storm can be predicted from the weather charts to arrive on the Norwegian coast on 11 September, and the swell is indeed observed at that time. Figure 7 shows the directional distribution at 0.07 Hz around the time when peak spectral energy was observed for that frequency band. The difference between the MLM and the MEM estimate is pronounced, with a circular spreading in the MLM estimate (60°) nearly twice that of the MEM estimate (35°). The somewhat large value of the spreading even for the MEM estimate is due to the general form of the distribution as a narrow spike superimposed upon a broad background. This form of distribution has been suggested previously (Barstow and Krogstad, 1984). The changes in swell direction may possibly be attributed to current refraction effects, although one must also take the sampling variability into account.

6. Discussion

The advantages of maximum entropy methods in spectral analysis over conventional FFT spectral analysis are substantial for short data series. In the present context this is carried over to the estimation of directional distributions by the correspondence treated in section 2. The MEM estimate has the favorable property of reproducing all Fourier coefficients which are input to the estimation. Moreover, the extrapolated higher-order Fourier coefficients are chosen in a way that may be said to be a most natural choice. For a heave/pitch/roll buoy, knowledge of c_1 and c_2 only greatly limits the resolving power for directionally adjacent peaks, but the MEM estimate is probably about the best that can be obtained unless further information on the directional distribution exists. It is, in principle, possible to include, for example, geographic constraints in the maximizing procedure, but only at the expense of the simplicity of the estimate. However, the fit of c_2 in addition to c_1 will discriminate between a variety of suggested distributions (cf. Barstow and Krogstad, 1983, 1984).

Acknowledgments. This manuscript has been prepared under the Analysis of Oceanographic Data (ANODA) research program funded by Conoco Norway, Inc., A/S Norske Shell, Elf Aquitaine Norge A/S, Total Marine Norsk A/S, Statoil, Saga Petroleum A/S, Norsk Hydro A/S and The Royal Norwegian Council for Scientific and Industrial Research. We greatly appreciate the program steering committee's permission to publish this material: C. Graham, A/S Norske Shell, G. Mallary, Conoco Norway Inc, E. Andersen, Elf Aquitaine Norge A/S, L. I. Eide, Norsk Hydro A/S, H. N. Lie, Saga Petroleum A/S, L. Staveland, Statoil, P. Antoine, Total Marine Norsk A/S.

We should also like to thank the sponsors of the ODAP program: Norsk Hydro A/S, Saga Petroleum A/S, Statoil, BP, Phillips Petroleum Co., SINTEF, IKU A/S, NTNF, Norsk Agip, Amoco, Arco Norway Inc., Conoco Norway Inc., Deminex (Norge) A/S, Elf Aquitaine Norge A/S, Esso Exploration and Production Norway Inc., A/S Norske Shell, Svenska Petroleum Exploration AB, Texaco Exploration Norway A/S, Total Marine Norsk A/S, Union Oil Norge A/S, Hispanoil, Volvo Energi AB.

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